B.Sc. Part II (Hons) 4th Paper LPDE

Q. $(D^3+1)y = \cos 2x$

Soln

For cf

$$D^3+1=0$$

$$\Rightarrow (D+1)(D^2-D+1)=0$$

$$\Rightarrow D+1=0 \quad \text{or} \quad D^2-D+1=0$$

$$D+1=0 \quad \Rightarrow D=-1$$

$$D^2-D+1=0 \Rightarrow D^2-2D\cdot\frac{1}{2}+\left(\frac{1}{2}\right)^2+1-\left(\frac{1}{2}\right)^2=0$$

$$\Rightarrow \left(D-\frac{1}{2}\right)^2 = \frac{-3}{4} = \frac{3i^2}{4}$$

$$\therefore D = \frac{1}{2} \pm \frac{\sqrt{3}}{2} i$$

$$\therefore D=-1,\ \frac{1}{2}+\frac{\sqrt{3}i}{2},\ \frac{1}{2}-\frac{\sqrt{3}i}{2}$$

$$\Rightarrow CF = Ae^{-x} + Be^{\left(\frac{1}{2}+\frac{\sqrt{3}i}{2}\right)x} + Ce^{\left(\frac{1}{2}-\frac{\sqrt{3}i}{2}\right)x}$$

$$= Ae^{-x} + e^{\frac{1}{2}x}\left(Be^{\frac{\sqrt{3}}{2}ix} + Ce^{-\frac{\sqrt{3}}{2}ix}\right)$$

$$= Ae^{-x} + e^{\frac{x}{2}}\left[B\left(\cos\frac{\sqrt{3}x}{2} + i\sin\frac{\sqrt{3}x}{2}\right) + C\left(\cos\frac{\sqrt{3}x}{2} - i\sin\frac{\sqrt{3}}{2}x\right)\right]$$

$$= Ae^{-x} + e^{\frac{x}{2}}\left[(B+C)\cos\frac{\sqrt{3}x}{2} + (B-C)i\sin\frac{\sqrt{3}}{2}\right]$$

$$\Rightarrow \quad CF = Ae^{-x} + e^{\frac{x}{2}}\left( D\cos\frac{\sqrt{3}x}{2} + E\sin\frac{\sqrt{3}x}{2}\right].$$

where D and E are constants.

Now, $PI = \dfrac{1}{D^3+1}\cos 2x = \dfrac{1}{D\cdot D^2+1}\cos 2x$

$$= \frac{1}{D\cdot(-2^2)+1}\cos 2x = \frac{1}{-4D+1}\cos 2x$$

$$= \frac{1}{1-4D}\cos 2x = \frac{1+4D}{(1-4D(1+4D)}\cos 2x$$

$$= \frac{(1+4D)}{1-16D^2}\cos 2x$$

$$= \frac{(1+4D)\cos 2x}{1-16\times(-2^2)} = \frac{(1+4D)\cos 2x}{1+64}$$

$$= \frac{1}{65}(1+4D)\cos 2x$$

$$= \frac{1}{65}\left[\cos 2x + 4D(\cos 2x)\right]$$

$$\Rightarrow PI = \frac{1}{65}\left[\cos 2x - 8\sin 2x\right]$$

Hence, complete solution is given by

$y = CF + PI$

$$\Rightarrow y = Ae^{-x} + e^{\frac{x}{2}}\left(D\cos\frac{\sqrt{3}x}{2} + E\sin\frac{\sqrt{3}x}{2}\right) + \frac{1}{65}(\cos 2x - 8\sin 2x)$$